

## The “three R’s” are not equal: Strengthening arithmetic capacities

Waldorf class teachers have exponentially more daily opportunities to take the pulse of each child’s language skill acquisition than is possible for math.



From the morning greeting and choral verse to the final handshake, almost every hour in the classroom will provide many moments for formative assessment of grammar, vocabulary, reading and writing. But the number sense and its progress usually

appear only during a fraction of the school day (and sometimes not every day).

This disparity of experience is probably even greater in most homes, because parents who regularly read to and then with their little children will easily form a general idea about progress. But especially in households or extended families where there is some skepticism about the Waldorf path to reading, daily story time might slide toward the dimension of analysis. Isn’t it true that, compared to arithmetic, there is much more parent awareness of and nervousness about reading in kindergarten and the younger grades?

In this article, I will try to present some ideas and suggestions for helping teachers (and parents) assess and support the foundational, pre-arithmetic skills that are essential to numeracy development in Kindergarten and the early grades. I’m hoping readers will try some of these for themselves, and then kindly email me feedback. Ideally the following might begin a shared research project.

Rudolf Steiner helped us consider an essential difference between reading/language development and arithmetic development when he pointed to language as conventional thinking, and math as sense-free thinking. That is, in whatever native language one resides, the names of things are evolving but agreed-upon indicators; say the word “chair” in English and an object with four legs is pretty much a given. Thus, within the context of a spoken phrase or sentence, surrounding objects and activities (referents) allow for daily addition of vocabulary and grammar. In a written sentence, a child learning to read can gain clues to unknown sight words from the surrounding known ones, as well as from pictures and the theme of the story.

In mathematics, however, referents and context are nearly absent, or require more mental will power to access. [1] As the grades curriculum progresses from manipulatives to arithmetic to math to algebra and beyond, students who have not mastered a preceding step are in great danger of forever losing the joy of numbers, self-defining as non-competent, and quietly shutting down... and

perhaps without the classroom acting out and attention seeking that students with reading difficulties often provide us as a ‘helpful hint’ to their need for extra support.

As noted in an excellent online article about early-grades numeracy foundations:

*“Children start elementary school with variable mathematics skills. Some children understand the fundamentals of numbers and mathematics, while others struggle with basic counting, number recognition, understanding of symbols, quantity discrimination, and concepts of addition and subtraction. Often, this set of early numerical competencies is referred to as number sense or early numeracy competencies. Students need to establish and understand these competencies before moving on to more complex mathematical tasks. This article describes important early numerical competencies and provides a description of how these competencies can be taught to students who struggle with mathematics.”*

“Being unable to grasp number concepts always induces tenderness and fear of the unknown...”  
Audrey McAllen in  
*The Extra Lesson* [2]

### Gender and learning-style differences

It is well known that the prevalence of dyslexia among boys is two to three times as high as it is for girls. What is less-well known is that there may also be gender differences in early maths learning styles and attitudes. Although most studies seem to

show that girls' achievement rates are equal to or slightly above those of boys, there can be attitudinal differences beneath the surface. According to an article published by the National Council of Teachers of Mathematics: *Research consistently shows that, even from a fairly young age, girls are less confident and more anxious about math than boys. Moreover, these differences in confidence and anxiety are larger than actual gender differences in math achievement. These attitudes are important predictors of math performance and math-related career choices... Some researchers have found that boys tend to use more novel problem-solving strategies, whereas girls are more likely to follow school-taught procedures. In general, girls more often follow teacher-given rules in the classroom, and it could be that this "good girl" tendency inhibits their math explorations and development of bold problem-solving skills. Such differences may contribute to gender gaps in mathematics as content becomes more complex and problem-solving situations call for more than learned procedures. Boys tend to be stronger in the ability to mentally represent and manipulate objects in space, and these skills predict better math performance and STEM career choices. Fortunately, some researchers have found that spatial skills can be improved through training, and one study even found that the gender gap in spatial skills was eliminated with training.* [3]

### Being at home in the house of numbers

People sometimes describe the Waldorf kindergarten as 'nonacademic,' but the truth is, with respect to language skills development, the Waldorf kindergarten strengthens key factors like auditory processing, comprehension, verbal recall, speech articulation, and an expansion of shared vocabulary... vital facets of language development that make the later steps in the academic process more readily achievable. Comprehension is built up in Kindergarten through the pictorial imagination necessary for reading comprehension in the grades and beyond.

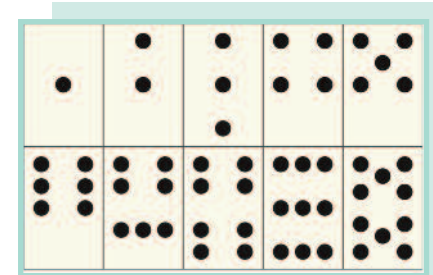
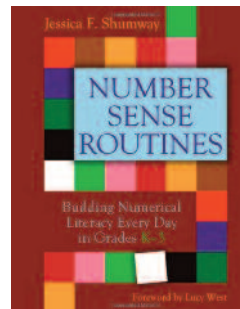
What, then, might be parallel factors for strengthening the young student's path to mathematics? Are there important accomplishments that need to come before computation – in the same way that, in the Waldorf approach, speech comes before writing and writing comes before reading? What is needed so that all students can reach that state described by many Waldorf teachers as "being at home in the house of numbers?"

### What comes before the four operations

In her book "Number Sense Routines", author Jessica Shumway sets out a clear and methodical approach to assessing and addressing pre-arithmetic skills in the classroom.

[4] An online review of this book notes:

*"Just as athletes stretch their muscles before every game and musicians play scales to keep their technique in tune, mathematical thinkers and problem solvers can benefit from daily warm-up exercises. Students with strong number sense understand numbers, ways to represent numbers, relationships among numbers, and number systems. They make reasonable estimates, compute fluently, use reasoning strategies (e.g., relate operations, such as addition and subtraction, to each other), and use visual models based on their number sense to solve problems. Students who never develop strong number sense will struggle with nearly all mathematical strands, from measurement and geometry to data and equations. Jessica Shumway has developed a series of routines designed to help young students internalize and deepen their facility with numbers. The daily use of these quick five-, ten-, or fifteen-minute experiences at the beginning of math class will help build students' number sense. The author shows that number sense can be taught to all students."*



Example:

### How important is subitizing?

An article on the website of the National Council of Teachers of Mathematics notes: "Three pictures hang in front of a six month-old child. The first shows two dots, the others show one dot and three dots. The infant hears three drumbeats. Her eyes move to the picture with three dots. Young children spontaneously use the ability to recognize and discriminate small numbers of objects. But some elementary school children cannot immediately name the number of pips showing on dice. What is this ability? When and how does it develop? Is it a special way of counting? Should we teach it?" The article later states: "Subitizing is a fundamental skill in the development of students' understanding of number" [5]

According to a German research paper, daily practice of subitizing and visual counting helped a group of students struggling in math to improve significantly. [6]

The material on this and the following pages summarizes the progression of ten mastery steps Ms. Shumway's extremely practical book delineates, and for which it provides many fun and useful activities.

**1. Subitizing** - able to perceive and accurately report small quantities of objects without counting.



*Part of a set of handmade cards that can be used to assess Magnitude*

**2. Magnitude** - able to tell you which of two sets of objects or dots has more without counting.

**3. Counting;** able to count accurately; counting on - i.e. from 1 to 50 by ones, with one-to-one correspondence (by end of kindergarten); from 20 to 60 without starting over; from 0 to 24 by twos.

**4. Ordinal & cardinal numbers** - able to place/find numbered objects in the correct order; to restate how many in a set of objects, without having to recount them.

**5. Hierarchical inclusion** - knows that a set of objects builds or reduces by one at a time, and that smaller numbers are part of bigger numbers

**6. Part-whole relationships** - once hierarchical inclusion is mastered, then able to progress to deconstruct component parts of a number, e.g. that 6 can be made up of 4 and 2, or 5 and 1.

**7. Compensation** - able to make trade-offs in part-whole relationships, e.g. if 7 and 3 equals 10, so does 6 and 4, because 6 is one less than 7, and 4 is one more than 3.

**8. Unitizing** - able to recognize how groups of numbers are constructed and patterned - able to separate a given quantity of manipulatives into simple unequal subgroups; into simple equal subgroups. Able to construct equal manipulative groups to total a given number, e.g. asked to construct "16" could make two groups of 8, or 4 groups of 4, etc.

**9. Number bonds** - has reached automaticity for all number bonds - for two single-digit numbers up to sums of 12; for two single-digit numbers up to sums of 20 (by end of first grade); for any combination of one and two digit numbers up to sums of 30. Knows times tables for all numbers up to 12 x 12.

**10. Four operations** - able to recognize what is being asked for with written and

word problems. Note that security/automaticity in all of the preceding nine steps are required before a student can begin arithmetic with ease and joy.

### Bibliography and notes

1. Going a little deeper in this vein, note that in a lecture on Sept. 9, 1922, Dr. Steiner stated: "Even with the most careful self-examination of which the soul is capable we cannot, by using only the capacities and powers of our ordinary consciousness, grasp the real nature of thinking and the formation of ideas."
2. "The Extra Lesson", by Audrey McAllen. Rudolf Steiner Press.
3. From "Current Research on Gender Differences in Math" by Colleen Ganley and Sarah Lubienski, posted May 9, 2016 <http://www.nctm.org/Publications/Teaching-Children-Mathematics/Blog/Current-Research-on-Gender-Differences-in-Math/>
4. "Number Sense Routines: Building Numerical Literacy Every Day in Grades K-3" by Jessica F. Shumway, ISBN-13: 978-1571107909
5. From an article posted on <http://www.nctm.org/> (full article available for purchase)
6. "Effects of Daily Practice on Subitizing, Visual Counting, and Basic Arithmetic Skills" available at <http://www.optomotorik.de/pubs/ovd39-2.pdf>  
See also "Subitizing: Vision Therapy for Math Deficits" at <http://www.lookingfor-learning.com/abstracts/40-4Art2.pdf>

The following are also highly recommended for further reading on the topic of math foundational considerations and approaches:

- "For Teachers: Conferences and Seminars on Arithmetic" by Karl König, at <http://www.waldorfresearchinstitute.org/pdf/Arithmetic.pdf>
- "Teaching Mathematics for First and Second Grade in Waldorf Schools" by Ernst Schuberth, available from Rudolf Steiner College Bookstore
- "Math Lessons for Elementary Grades" by Dorothy Harrer, ISBN-13: 978-1888365498
- "Diagnosis and Remedial Teaching in Arithmetic" by Fred J. Schonell and F. Eleanor Schonell, ISBN-13: 978-0050003459 (also on this website at [http://www.movementforchildhood.com/uploads/2/1/6/7/21671438/schonell\\_math.pdf](http://www.movementforchildhood.com/uploads/2/1/6/7/21671438/schonell_math.pdf))
- "Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States" by Liping Ma, ISBN-13: 978-0415873840
- "Active Arithmetic!: Movement and Mathematics Teaching in the Lower Grades of a Waldorf School" by Henning Andersen, AWSNA Publications
- Boys and Girls in Movement - [www.movementforchildhood.com/uploads/2/1/6/7/21671438/boysandgirls.pdf](http://www.movementforchildhood.com/uploads/2/1/6/7/21671438/boysandgirls.pdf)

**Table 1: Early Childhood and Primary Grades Number Sense Strengthening & Screening**

*A step-by-step path to building the foundations for success – and for identifying gaps that will forever limit success*

Step	By Gr.	Description	Activities for Learning & Observation
<b>Subitizing</b>	K	Able to perceive and accurately report small quantities of objects without counting	Dot cards, dominoes, dice
<b>Magnitude</b>	K-1	Able to tell which of two sets of objects or dots has more without counting	Two-sided colored-dot flash cards; number guessing game
<b>Counting</b> Able to count accurately; to count on	K	From 1 to 50 by ones, with one-to-one correspondence (by end of KG)	Finger holding game, object counting, drum beats, board games, interview, counting frame, number line. Object counting on with ability to restate quantity after a pause or distraction
	1.5	From 20 to 60 without starting over	
	1.5	From 0 to 24 by twos	
<b>Ordinal &amp; cardinal numbers</b>	1.0	Able to place/find numbered objects in the correct order; to restate how many in a set of objects, without having to recount them	See Table 2
<b>Hierarchical inclusion</b>	1.0	Knows that a set of objects builds or reduces by one at a time, and that smaller numbers are part of bigger numbers	Object counting with ability to recognize moving part of the group leaves total intact
<b>Part-whole relationships</b>	1.5	Once hierarchical inclusion is mastered, then able to progress to consider parts of a number, e.g. that 6 can be made up of 4 and 2, or 5 and 1	Object counting with addition to/subtraction from a previously counted group
<b>Compensation</b>	1.5	Able to make trade-offs in part-whole relationships, e.g. if 7 and 3 equals 10, so does 6 and 4, because 6 is one less than 7, and 4 is one more than 3	See Table 3
<b>Unitizing</b> Able to recognize how groups of numbers are constructed and patterned	1.5	Able to separate a given quantity of manipulatives into simple unequal subgroups	See Table 3
	2.0	Able to separate a given quantity of manipulatives into simple equal subgroups	
	2.5	Able to construct equal manipulative groups to total a given number, e.g. asked to construct "16" could make two groups of 8, or 4 groups of 4, etc.	
<b>Number bonds</b> Has reached automaticity for all number bonds	1.5	For two single-digit numbers up to sums of 12	Dot cards, flash cards, counting frame, number line, worksheets
	2.0	For two single-digit numbers up to sums of 20 (by end of first grade)	
	3.5	For any combination of one and two digit numbers up to sums of 30	Flash cards, number lines
	4.0	Knows times tables for all numbers up to 12 x 12	Rhythmic counting, flash cards, daily worksheets
<b>4 Operations</b>	1-3	Able to recognize what is being asked for with written and word problems	See Table 2

The above progression of capacities is based primarily on the book *Number Sense Routines* by Jessica F. Shumway

**Table 2: K- 1 Number Sense Screening Record** • Student \_\_\_\_\_ Date \_\_\_\_\_ Teacher \_\_\_\_\_

Step	Activities	Observations
<b>Subitizing</b>	<ol style="list-style-type: none"> <li>1. Arrange deck of 10 dot cards in a row</li> <li>2. Point to each and count them off, forward and back</li> <li>3. Visually match card flashed by teacher</li> <li>4. "Flash card" activity - if student can name one card at a time, then check for two cards side by side (for a total).</li> </ol>	
<b>Magnitude</b>	<ol style="list-style-type: none"> <li>1. Blue &amp; red dot cards</li> <li>2. Number line - have child point on a number, then ask for "three bigger", "two smaller", etc.</li> </ol>	
<b>Counting</b>	<ol style="list-style-type: none"> <li>1. Verbally from 1 to 30 by ones; and from 20 to 40 without starting over. Then from 0 to 24 by twos.</li> <li>2. Number line - have student point to 6, then continue to 20.</li> <li>3. Beads - estimate how many red beads in a bowl, then count them into a smaller bowl placed on the right. Then estimate how many blue beads, and count them into a bowl placed to the left. Observe hand use as well as accuracy.</li> </ol>	
<b>Ordinal &amp; cardinal numbers</b>	<ol style="list-style-type: none"> <li>1. Set out a row of 12 numbered cups. Ask student to find the 5th cup, the 3rd, etc. as ability allows.</li> <li>2a. Set out a row of 7 blocks, have student count them out loud, then say "Let's remember how many."</li> <li>2b. After next activity, come back to them and say "I'm going to add n more; now how many do we have?"</li> <li>3. Have student turn around, and hide an object under one of the cups. Then ask for a guess which number cup it's hiding under. Provide clues re: higher or lower and observe until found. Repeat once more.</li> </ol>	
<b>Hierarchical inclusion and part-whole relationships</b>	<p>Use objects in #2 above.</p> <ol style="list-style-type: none"> <li>1. Add or take away one or two objects and ask what is there. Also ask "how many more/less are needed for n?"</li> <li>2. Ask student to give you three, give you five etc. Observe that he/she does not confuse this with giving you the third or fifth.</li> </ol>	

**Table 3: Grade 1 - 2 - 3 Number Sense Screening Record** • Student \_\_\_\_\_ Date \_\_\_\_\_ Teacher \_\_\_\_\_

Step	Activities	Observations
<b>Compensation</b>	Able to make trade-offs in part-whole relationships, e.g. if 7 and 3 equals 10, so does 6 and 4, because 6 is one less than 7, and 4 is one more than 3.	
<b>Unitizing</b> Able to recognize how groups of numbers are constructed and patterned	Able to separate a given quantity of manipulatives into simple unequal subgroups	
	Able to separate a given quantity of manipulatives into simple equal subgroups	
	Able to construct equal manipulative groups to total a given number, e.g asked to construct "16" could make two groups of 8, or 4 groups of 4, etc.	
<b>Number bonds</b> Has reached automaticity for all number bonds	For two single-digit numbers up to sums of 12	
	For two single-digit numbers up to sums of 20 (by end of first grade)	
	For any combination of one and two digit numbers up to sums of 30	
	Knows times tables for all numbers up to 12 x 12	
<b>Four operations</b> Able to recognize what is being asked for with written and word problems	Addition	
	Subtraction	
	Multiplication	
	Division	

## "Math Explorers Club" - a starter kit of whole-class activities for math foundations

### Activities to introduce in Kindergarten or Grade 1

#### Blindfold counting with feet

- Goal Play
- Supports Inner picturing and movement
- Materials Blindfolds, stones
- Description Work in pairs, one child blindfolded in chair, other seated on floor. Seated child sets out a pile of 10 to 20 stones, for the other to count using feet.

#### Dot card addition

- Goal Consolidate number bonds
- Supports Subitizing; counting; arithmetic
- Materials Dot cards with numbered backs - one set per student. Teacher also to have a set.
- Description At desks. Have students sort their decks from 1 to 10 in a straight/side-by-side row at the top of desk space. Teacher holds up a dot card, and has students place their (same) card in middle of desk space. Then, teacher says: "What card do we need to get to x?" (i.e. a five card is out, and the total needed is nine; therefore a four card is to be found and placed next to the 5-dot card.) Then, students can be asked to turn these two cards over to the numbered side and recite, i.e. "4 plus five or five plus four equals nine."

#### Dot card concentration

- Goal Play
- Supports Subitizing; spatial intelligence
- Materials Dot cards with blank backs - two or four sets per playing group.
- Description Children can play in groups of two or three. Shuffle all cards and then put them face up in a grid arrangement (if 2 sets/20 cards, then 4 x 5 grid; if 4 sets/40 cards then 5 x 8). After cards arranged and studied for a minute, turn them all over - face down. The students in turn try to find pairs and if successful pick up and add to his/her pile. Winner is one who finds the most pairs. Can be played either as single-pick turn, or as a player keeps picking as long as he/she finds a pair.

#### Dot card number bonds

- Goal Consolidate number bonds
- Supports Subitizing; counting on or off; arithmetic
- Materials Dot cards with numbered backs - one set per student. Teacher also to have a set.
- Description At desks. Have students sort their decks from 1 to 10 in a straight/side-by-side row at the top of desk space. Teacher calls out a number and asks students to set out two cards that add up to that number. When students have completed this task, then call on a few students to describe the pairings they found.

#### Fingers behind back addition

- Goal Inner picturing
- Supports Digit awareness; number bonds
- Materials None
- Description Teacher stands behind student; student puts both hands behind back with fingers loosely extended. Teacher gently grabs/holds a combination of fingers from each hand (e.g. 3 fingers on the right and two on the left) and asks the student "How many?"

### Activities to introduce in Grade 2

#### Double Digit War - Card Game

- Goal Play
- Supports Place value, bigger/smaller
- Materials One deck of cards for each pair of players, with face cards removed; one sheet of paper for each player, marked & lined for tens and ones column
- Description Groups of two: Players shuffle and split deck so each has a half-deck pile. Game is played just like regular "War" (i.e. higher number takes both cards) except that each player looks privately at his/her first card and then in unison they slap their first cards onto their respective mats, choosing to slap it into the tens column or the ones column. Then, more or less in unison they take the second card and place it in the open column.

#### Playing card concentration

- Goal Play
- Supports Memory; spatial intelligence
- Materials Playing cards - one deck per playing group.
- Description Children can play in groups of two or three. Adjust number of cards to be used for group size and ability level, i.e. perhaps remove all face cards. Shuffle all cards and then put them face up in a grid arrangement. After cards arranged and studied for a minute, turn them all over - face down. The students in turn try to find pairs (either any two of a number, or two matching color numbers) and if successful pick up and add to his/her pile. Winner is one who finds the most pairs. Can be played either as single-pick turn, or as a player keeps picking as long as he/she finds a pair.

#### Math strengthening activities from *The Extra Lesson*

All exercises for Body Geography

Tracing hands (and feet)

Form drawings: Counting Star; Interpenetrating Triangles; Expanding and Contracting Triangles